Piecewise-linear Modeling of Multivariate Geometric Extremes

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Limit sets and gauge functions

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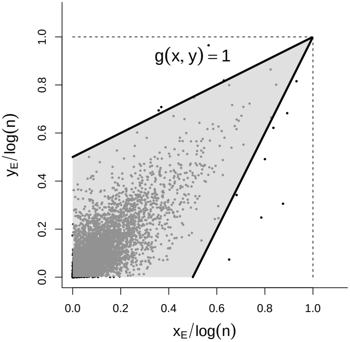
  where the **gauge function**, $g : \mathbb{R}^d \longrightarrow \mathbb{R}_+$, is 1-homogeneous and continuous...
- ...*then* scaled sample clouds $\left\{\frac{\boldsymbol{X}_1}{\log n}, \ldots, \frac{\boldsymbol{X}_n}{\log n}\right\}$ converge onto a **limit set**,

$$G := \left\{\boldsymbol{x} \in \mathbb{R}^d \,\middle|\, g(\boldsymbol{x}) \leq 1\right\}$$

  as $n \to \infty$ (Balkema and Nolde, 2010; Nolde and Wadsworth, 2022).

Limit sets and gauge functions



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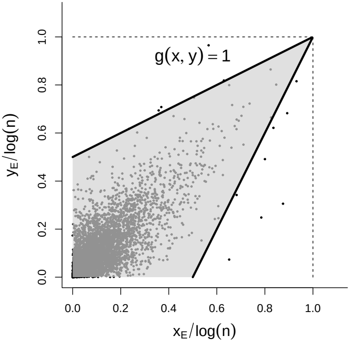
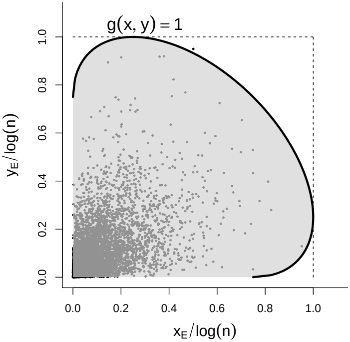
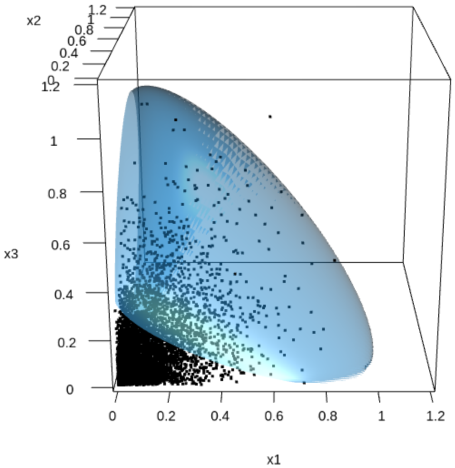




“pointy” $\Longrightarrow$ AD

“not pointy” $\Longrightarrow$ AI

Limit sets and gauge functions



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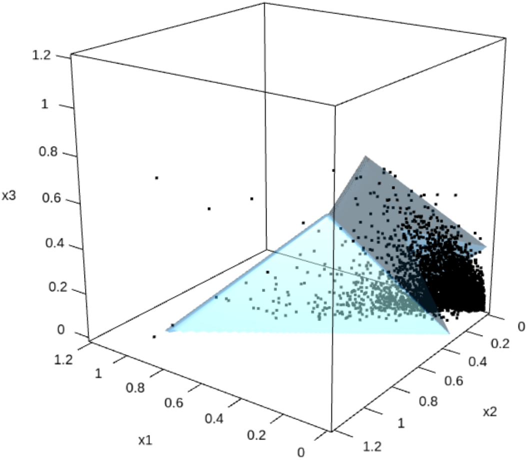

Limit sets and gauge functions

Is this approach useful for extremal statistical inference across the multivariate tail?

If so, how can we estimate g from data and use it for extremal statistical inference?

Extremal inference with gauge functions

- Define $(R, \boldsymbol{W}) = (\|\boldsymbol{X}\|_1, \boldsymbol{X}/\|\boldsymbol{X}\|_1)$
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- **Goal:** Model for $R \mid \{\boldsymbol{W} = \boldsymbol{w}, R > r_q(\boldsymbol{W})\}$.

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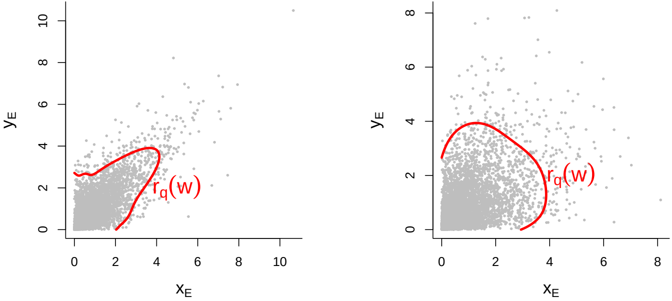

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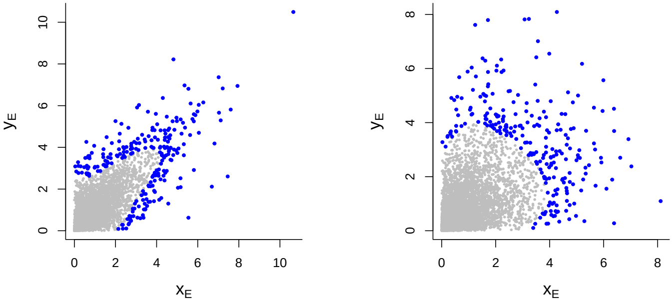

Geometric approach: model fitting

- Wadsworth and Campbell (2024): fit the model

$$R \mid \{\boldsymbol{W} = \boldsymbol{w}, R > r_q(\boldsymbol{w})\} \sim \text{truncGamma}(\alpha, g(\boldsymbol{w}; \boldsymbol{\theta}))$$

by maximizing

$$L(\boldsymbol{\theta}; r_{1:n}, \boldsymbol{w}_{1:n}) = \prod_{i:r_i > r_q(\boldsymbol{w}_i)} \frac{f_{\text{Gamma}}\left(r_i; \alpha, g(\boldsymbol{w}_i; \boldsymbol{\theta})\right)}{\bar{F}_{\text{Gamma}}\left(r_q(\boldsymbol{w}_i); \alpha, g(\boldsymbol{w}_i; \boldsymbol{\theta})\right)}$$

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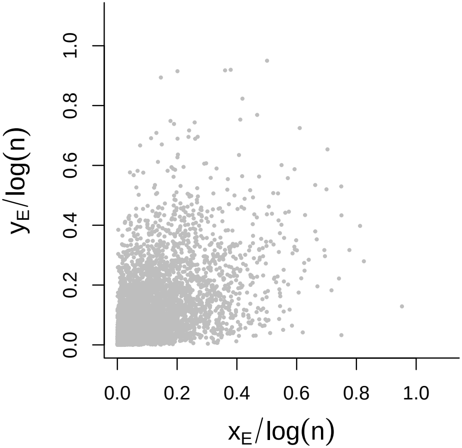
- **Campbell and Wadsworth (2024): Define $g(\boldsymbol{w}; \boldsymbol{\theta})$ piecewise-linearly.**

Semiparametric piecewise-linear approach



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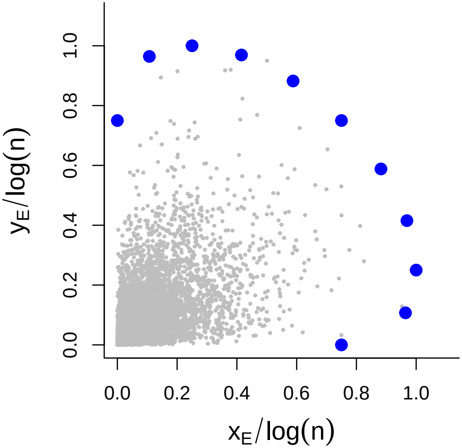

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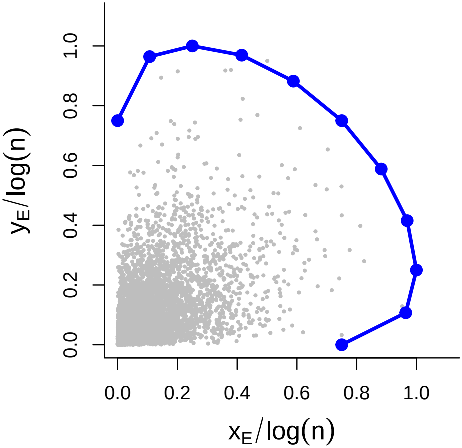

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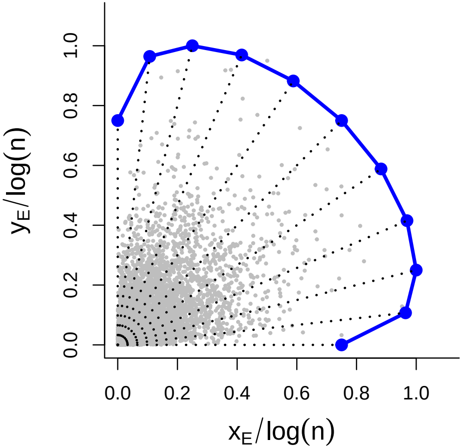

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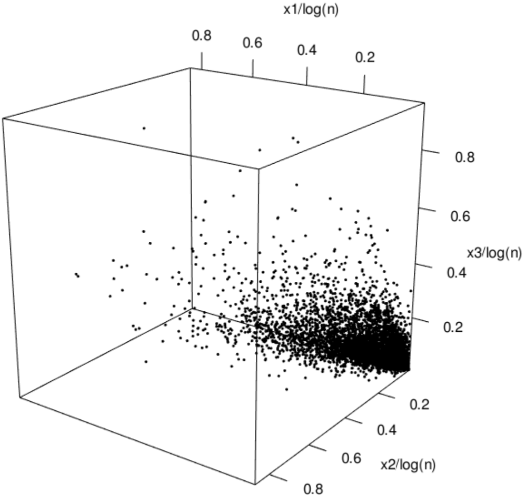

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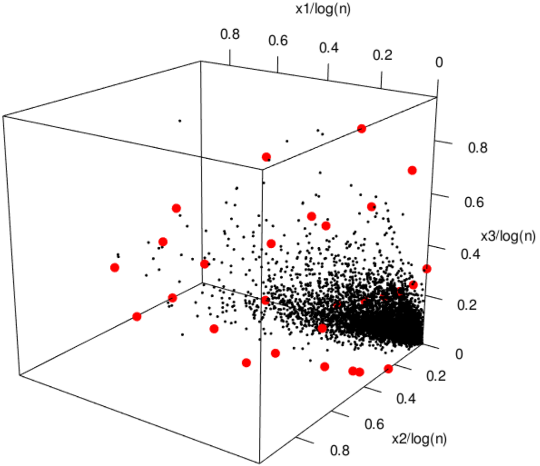

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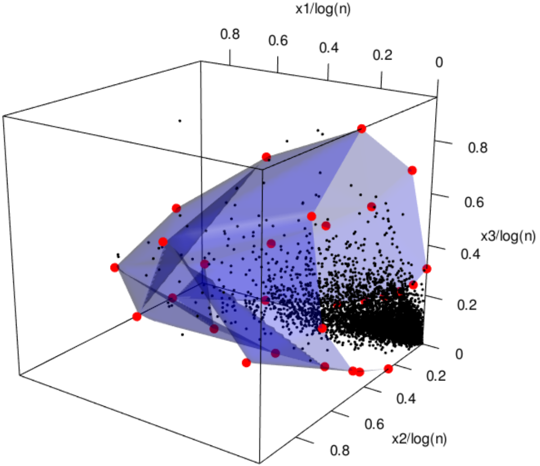

Semiparametric piecewise-linear approach

In d-dimensions...

- Define a set of N reference angles $\boldsymbol{w}^{\star 1}, \ldots, \boldsymbol{w}^{\star N} \in \mathcal{S}_{d-1}$.
- Results in M regions: $\triangle^{(1)}, \triangle^{(2)}, \ldots, \triangle^{(M)}$.

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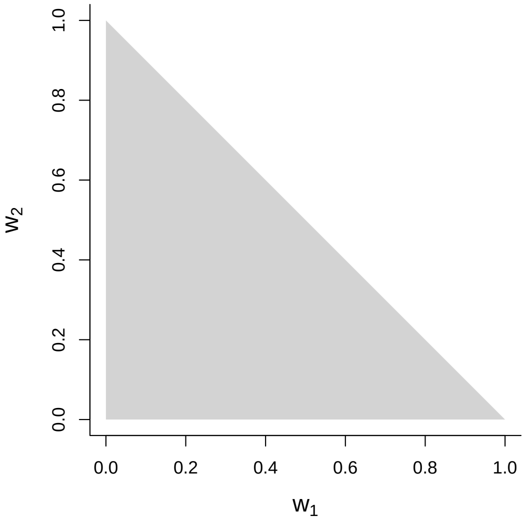
- At a point $\boldsymbol{x} \in \mathbb{R}^d$, the gauge function value is given by

$$g_{\mathrm{PWL}}(\boldsymbol{x}; \boldsymbol{\theta}) = \sum_{k=1}^{M} \mathbf{1}_{\triangle^{(k)}}\left(\boldsymbol{x}/\|\boldsymbol{x}\|\right) \frac{\boldsymbol{n}^{(k)\top}\boldsymbol{x}}{\boldsymbol{n}^{(k)\top}\theta_1^{(k)}\boldsymbol{w}^{\star(k),1}}$$

Semiparametric piecewise-linear approach

Partitioning the angular space, $\mathcal{S}_2 = \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\|_1 = 1\}$



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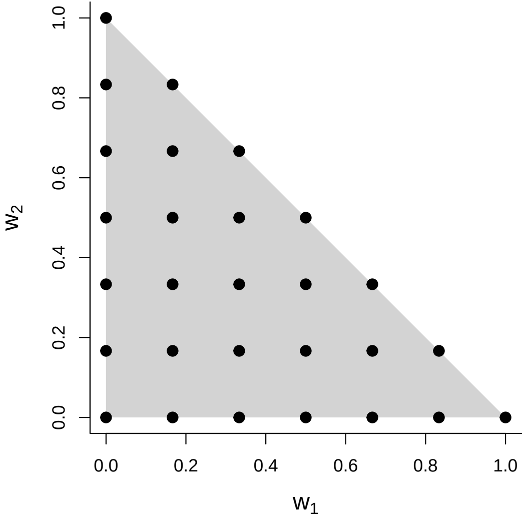

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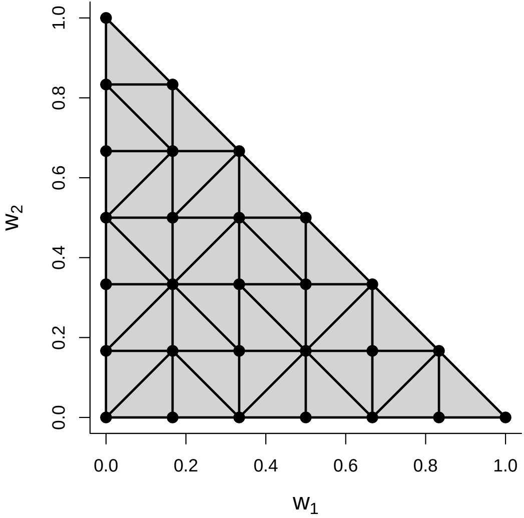

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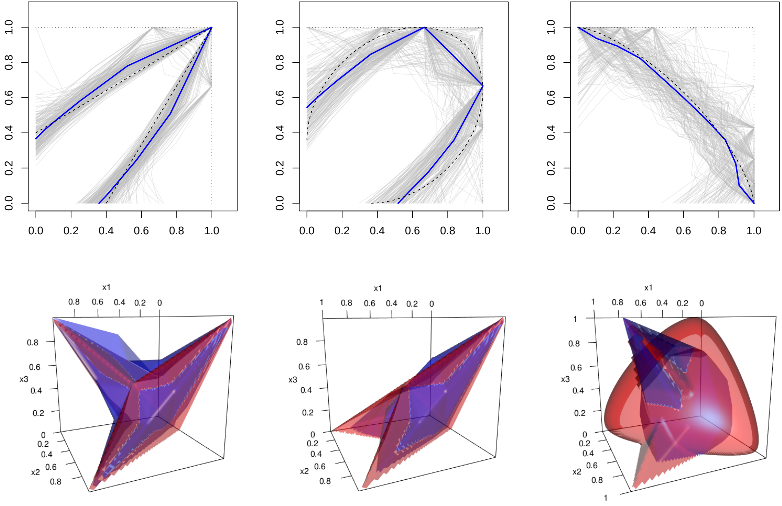

Geometric approach: extrapolation

- Can estimate extremal probabilities via

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- Compute $\Pr\left(\boldsymbol{X} \in B | R > r_q(\boldsymbol{W})\right)$ via sampling:
  1. sample $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_N$ from $\boldsymbol{W} \mid \{R > r_q(\boldsymbol{W})\}$.
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- Compute $\Pr\left(R > r_q(\boldsymbol{W})\right)$ empirically.

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- Compute $\Pr(R > r_q(\boldsymbol{W}))$ empirically.
- **Bonus!** Can extrapolate far into the tails using

$$\Pr(\boldsymbol{X} \in B) = \Pr(\boldsymbol{X} \in B | R > k r_q(\boldsymbol{W})) \Pr(R > k r_q(\boldsymbol{W}))$$

for $k > 1$.

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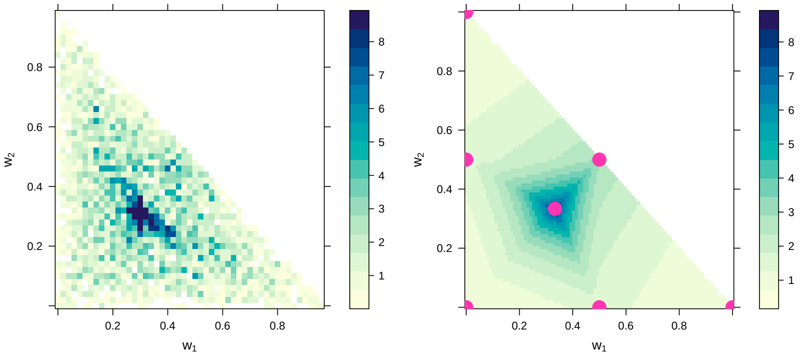
- Campbell and Wadsworth (2024):
  - $\mathrm{vol}(\{\boldsymbol{x} : g_{\mathrm{PWL}}(\boldsymbol{x};\boldsymbol{\theta}) \leq 1\})$ has a closed-form expression.
  - samples are drawn from $f_{\boldsymbol{W}\mid\{R>r_q(\boldsymbol{W})\}}$ using MCMC.

Angular model

Fitting the angular density

$$f_{\boldsymbol{W}|\{R>r_q(\boldsymbol{W})\}}(\boldsymbol{w};\boldsymbol{\theta}) = \frac{g_{\mathrm{PWL}}(\boldsymbol{w};\boldsymbol{\theta})^{-d}}{d\mathrm{vol}(\{\boldsymbol{x} : g_{\mathrm{PWL}}(\boldsymbol{x};\boldsymbol{\theta}) \leq 1\})}$$



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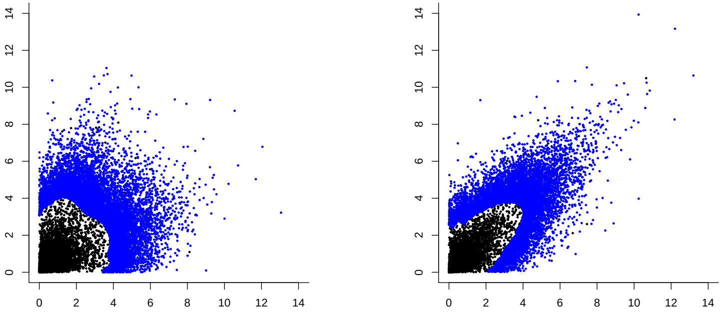

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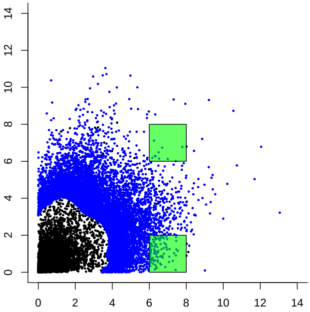

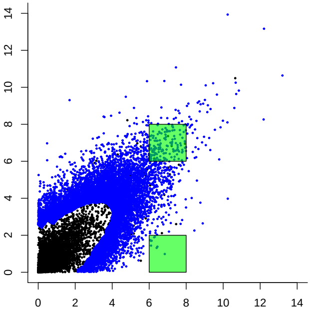

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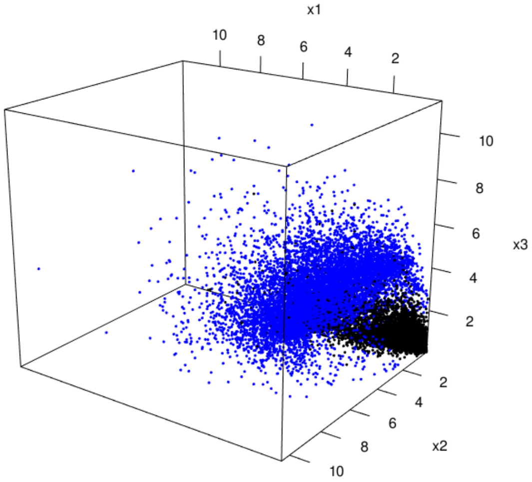

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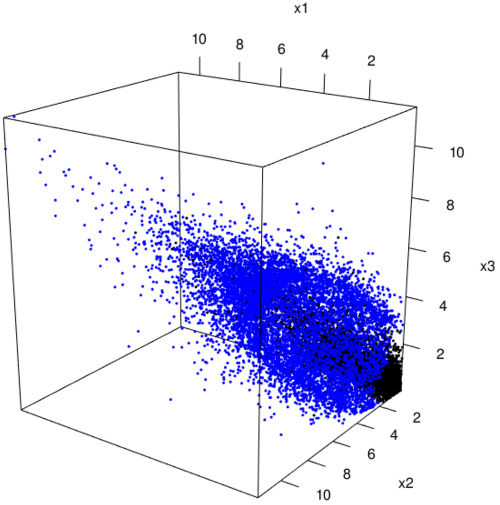

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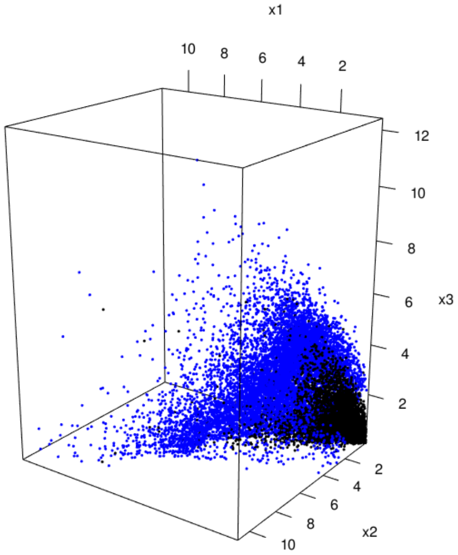

Application to air pollution measurements ($d = 3$)

- North Kensington site, London, UK
- carbon monoxide (CO, mg/m^3), nitrogen dioxide (NO_2, $\mu\text{g/m}^3$), and particles with a diameter of 10 μm or less (PM10, mg/m^3).
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- $n = 5,584$ daily maximum measurements, October–April. 1996–2024.
- Using methods from Simpson et al. (2020), evidence that
  - $\{\text{PM10}\}$ is large when $\{\text{CO}, \text{NO}_2\}$ are small
  - $\{\text{CO}, \text{NO}_2, \text{PM10}\}$ grow large together.

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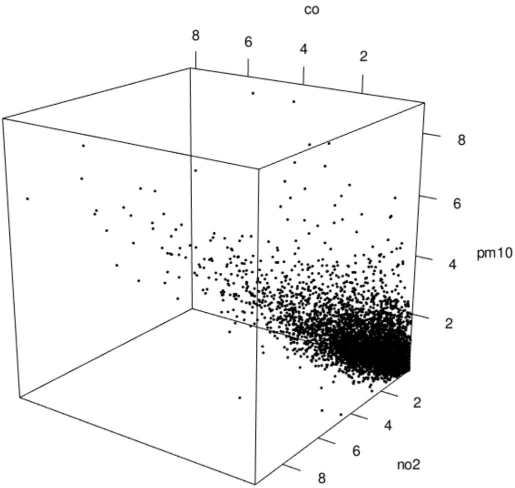

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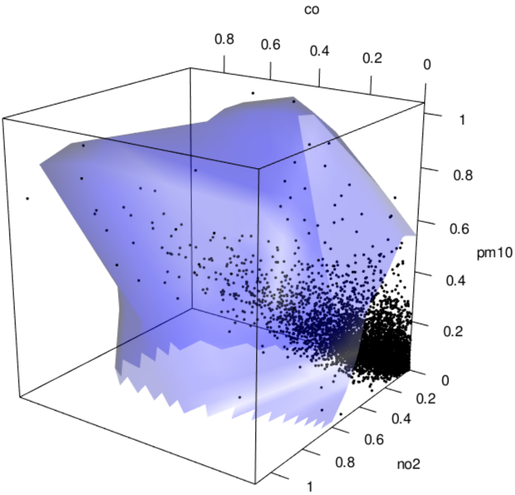
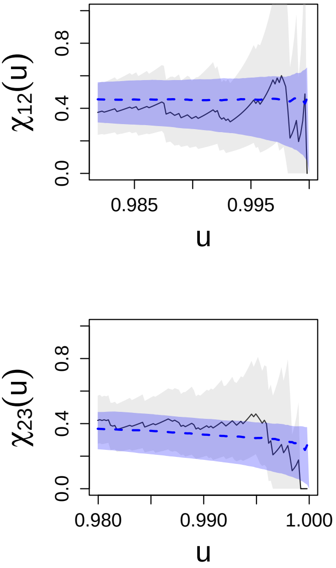


$\chi_C(u)$ estimates, $C \subseteq \{1, 2, 3\}$

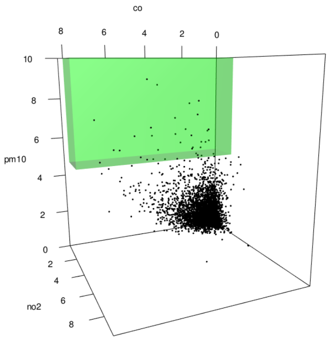


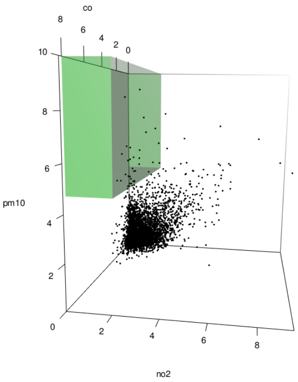
An alternate tail probability

For $d = 3$ pollution data, consider

$$\psi_{\{3\}}(u; \delta_1, \delta_2) = \Pr\left[F_E(X_1) < \delta_1, F_E(X_2) < \delta_2, F_E(X_3) > u\right]$$

for $\delta_1, \delta_2 \in [0, 1)$, $u > u_0$, $u_0 \in [0, 1)$ close to 1.





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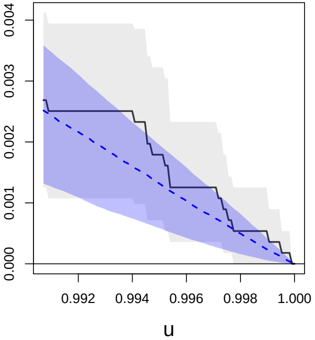
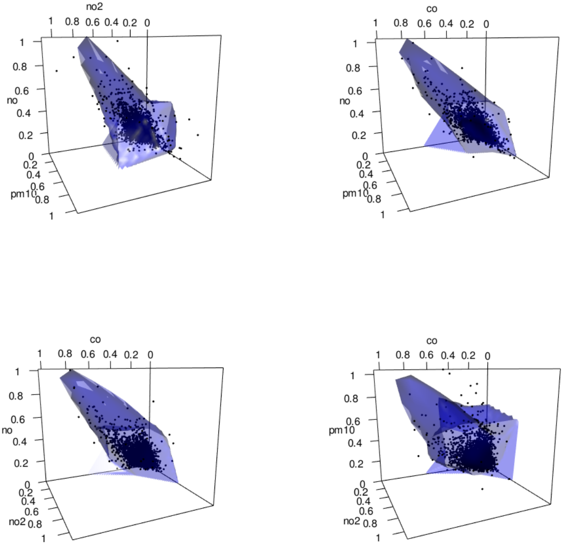


Application to air pollution measurements ($d = 4$)



Thank you!

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